Exam Analysis 2013

Date : 02-12-2013
Time : 18.30 - 21.30
Place : Aletta Jacobshal 01

Provide clear arguments for all your answers; 'yes' or 'no' answers are not allowed. In your argumentation you may use all theorems and statements in the book. However, you should indicate which theorems/ statements you are using.

The detailed grading scheme can be found below.

1. Assume that $\lim_{x\to c} f(x) = L$, where $L \neq 0$, and assume that $\lim_{x\to c} g(x) = 0$. Prove that
$$\lim_{x\to c} \frac{|f(x)|}{|g(x)|} = \infty$$

2. Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume that $f'(x) \to 0$ for $x \to \infty$. Prove that the function $g : \mathbb{R} \to \mathbb{R}$ defined by
$$g(x) = f(x+1) - f(x)$$
satisfies
$$g(x) \to 0 \text{ for } x \to \infty.$$

3. Consider a non-empty subset $A \subset \mathbb{R}$. Assume that A is both open and closed. Prove by direct arguments that A is not bounded.

4. Consider a non-empty subset $A \subset \mathbb{R}$. Give the definition of a limit point of A. Prove that the set of limit points of A is a closed set.

5. Determine the pointwise limit of the following sequences of functions f_n, $n \in \mathbb{N}$, and prove or disprove their uniform convergence:
   (a) $f_n(x) = 0$, $x \leq n$, and $f_n(x) = x - n$, $x \geq n$, on $(0, \infty)$;
   (b) $f_n(x) = e^{-nx^2}$ op $[-1, 1]$;
   (c) $f_n(x) = \frac{e^{-x^2}}{n}$ op $[-1, 1]$.

6. (a) Prove that $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{2^n}$ is continuous on $\mathbb{R}$.
   (b) Prove that $h(x) = \sum_{n=1}^{\infty} \frac{x}{n^2}$ is continuous on $\mathbb{R}$. (**Hint**; First prove continuity on an interval $[-a, a], a > 0$.)
   (c) Prove that $h(x)$ defined in the previous part is differentiable on $\mathbb{R}$. Prove that $h'(x)$ is a constant function.

7. Define the function $h : [0, 2] \to \mathbb{R}$ as

   $h(x) = 1, \text{ if } x \neq 1, \quad h(1) = 0.$

   Argue that h is integrable.
   Define $H(x) = \int_0^x h$ for each $x \in [0, 2]$. Show that H is differentiable at every point of $[0, 2]$. What is $H'(x)$, $x \in [0, 2]$?

Grading scheme:

Total 100, Free 10.

1. 15.
2. 12.
3. 10.
4. 10.
5. a: 5, b: 5, c: 5.
6. a: 5, b: 6, c: 7.
7. 10.